CFD 1.0

- 1D Linear Convection
- 1D Non-Linear Convection
- 1D Diffusion

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Book

- Numerical Heat Transfer and fluid flow — Patankar

  ⇓

  finite volume method

- Anderson CFD and Heat Transfer } 1st book

- Malala Sekhara

- Moukaled Darwish, CFD (FVM)

Computational Fluid Dynamics.

CFD → Tool

Experimental vs. Analytical vs. Numerical.

actual physics → Experimental

actual solution → Analytical

Numerical → Not real

experimental not possible because high cost and damage, and analytical cannot be applied in every complex situation.

① Calculus → integral, differential
↓
② numerical method
↓
③ CFD (theory)
↓
④ in-house Code
↓
⑤ Software

$\boxed{\text{Numerical methods}}$ → many types of simulations

↓

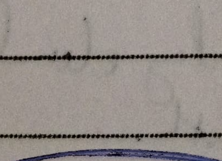
here, we focus at CFD → many methods. (depending on method of discretisation)

- → FDM
- → FEM (finite element method)
- → FVM
- → C.V.FEM (Control volume FEM)
- → BEM

↓

we'll do Finite difference Method.

Discretisation → not all eq^n's are solvable on complete domain



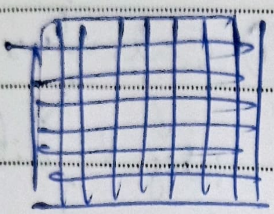
discretisation (dividing in parts)

→ analytical method (Double differential method)

Governing eq^n

Partial differential eq^n $\Longrightarrow$ Algebraic eq^n

Discretisation
(x, y, z, t)



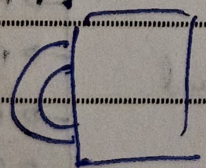
Mesh/grid

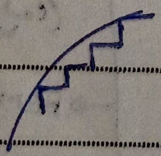
different solution

Solution $\longleftarrow$ Merge

1. Pre processing
2. Solution
3. Post processing

Finite difference method
$\Downarrow$





Finite volume
$\Downarrow$
many control volume.

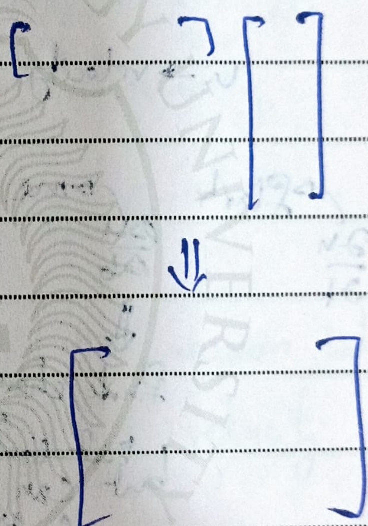
meshless
$\Downarrow$
boundary element, smooth.

Lattice boltzmann!!

Course Objectives

1)
2)
3)
4)
5)
6)

two linspace (1D) arrays multiplied makes a square array.



(unsteady)

1-D Linear Convection

Convect → to move

linear → uniform

x, y, z, t — 1-space dimension.

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0}$$

u → velocity

$\frac{\partial u}{\partial t}$ → flash

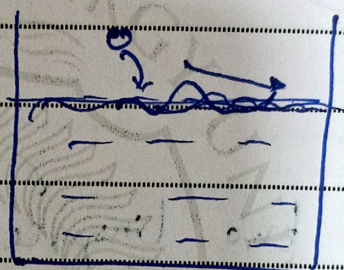
$\frac{\Delta u}{\Delta t}$ → me

⇓

finite change

but ~~[illegible]~~ can be continuous

→ so it can be written as $\frac{\partial u}{\partial t}$



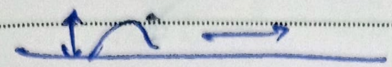
The wave is created on the surface of water and moves after a disturbance.

Continuum Space-time.

→ everything is happening smoothly, no singularity

$$\lim_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = \frac{du}{dt} \text{ or } \frac{\partial u}{\partial t}$$

infinitesimally small change of velocity per infinitesimally small change in time



$\frac{du}{dx}$;

$\frac{d^2u}{dx^2}$ $\Rightarrow$ effect of change in this dimension on other.

Like here, diffusion is also happening, the amplitude of wave is also decreasing.

$\frac{\partial u}{\partial x}$ $\rightarrow$ gradient w.r.t. space.

$\hookrightarrow$ this is the reason behind change of velocity.

What are we modelling?

$\downarrow$

$u = f(x, t)$ to decide the governing eqn.

1D Linear steady state Convection

$\Downarrow$

$\frac{\partial u}{\partial t} = 0$, $c\frac{\partial u}{\partial x} = 0$ $\rightarrow$ $\frac{du}{dx} = 0$

2D

$\hookrightarrow$ $\frac{\partial u}{\partial t} + c_1\frac{\partial u}{\partial x} + c_2\frac{\partial u}{\partial y} = 0$

$$\frac{\partial u}{\partial t} \Rightarrow \cancel{\lim_{\Delta t \to 0}} \frac{\Delta u}{\Delta t}$$

PDE → algebraic

$$\frac{\partial u}{\partial t} \approx \frac{\Delta u}{\Delta t}$$

→ approximation

⇓

We $\boxed{\text{validate}}$ with experimental or analytical data

error < 8%

$\frac{\Delta u}{\Delta t}$ is algebraic

→ $\frac{u_2 - u_1}{t_2 - t_1}$

Validation to nearest problem

Discretization of eq^{n}s.

$\boxed{\text{Forward scheme}}$ as time always moves forward

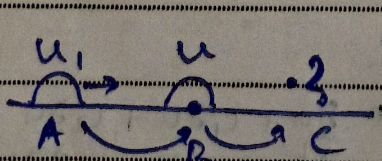
↓

one-way coordinate

$$\frac{\partial u}{\partial t} \approx \frac{\Delta u}{\Delta t} \rightarrow \frac{u' - u}{\Delta t}$$

$$\frac{\partial u}{\partial x} \approx \frac{\Delta u}{\Delta x} \rightarrow \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

→ x is two-way coordinate



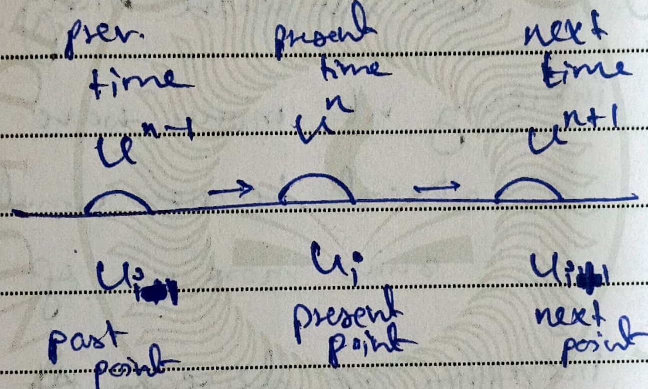
u depends on point A but can depend on future point C only sudden change (like pressure gradient present).

Eg (Forward scheme or Central scheme)

$$\therefore \frac{\partial u}{\partial t} = \frac{u_i^{n+1} - u_i^{n}}{\Delta t}$$

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^{n} - u_i^{n}}{\Delta x}$$

$$\therefore \frac{u_i^{n+1} - u_i^{n}}{\Delta t} = -c\left[\frac{u_{i+1}^{n} - u_i^{n}}{\Delta x}\right]$$



$n \rightarrow$ time

$i \rightarrow$ space

$$\therefore \boxed{u_i^{n+1} = u_i^{n} - \Delta t \cdot c\left[\frac{u_{i+1}^{n} - u_i^{n}}{\Delta x}\right]}$$

new value = old value + $\Delta t \times$ slope

→ the present can be only calculated from past and similarly, → future from present and this can be done for so on.

→ Initial Conditions
→ Boundary Conditions.

in the code,

→ increasing nx increases accuracy

↳ But after one limit, graphs does not behave nicely. → instability in simulation

→ increasing nt moves wave more ahead because simulation runs more no. of times.

→ increasing or decreasing dt a lot also gives distorted output

↳ instability in simulation

To deal with instability in the simulation,

↳ Courant Number (Co)

$$\boxed{Co = \frac{\overbrace{u}^{\text{max } u} \cdot \Delta t}{\Delta x}} \quad (\sigma)$$

$Co < 1$ → for reliable results.

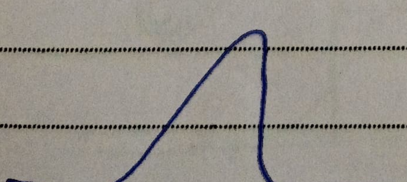
Linear → $c \dfrac{\partial u}{\partial x}$ → velocity gradient moving with velocity c.

$u \dfrac{\partial u}{\partial x}$ → velocity gradient moving with velocity u, which in turn is changing

divergence

$$u \cdot \nabla = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

non linear with forward difference scheme,



non linear with central scheme will have both sides non-linear

1D Nonlinear Convection

for non-linear,

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

$$\cancel{\therefore \frac{u_i^{n+1} - u_i^n}{\Delta t} = u_i^n \left[\frac{u_{i+1}^n - u_i^n}{\Delta x}\right]}$$

$$\cancel{\therefore u_i^{n+1} - u_i^n = \Delta t \cdot u_i^n \cdot \left[u_i^{n+1}\right.}$$

$$\therefore \frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\frac{u_{i+1}^n - u_i^n}{\Delta x}\right] = 0$$

$$\boxed{\therefore u_i^{n+1} = u_i^n + u_i^n \times \frac{\Delta t}{\Delta x} \times \left[u_{i+1}^n - u_i^n\right]}$$

Convergence and Courant No.

→ which result is right?

→ never use random nos.

→ Grid Dependence Test

→ Time Dependence Test

Unsteady simulation → independent of choice of size of grid

→ result should also not vary with Δt.

Courant Number

Courant - Fredrick - Levi (CFL) Condition flux ← (or) → velocity - far field

highest velocity value

$$\boxed{C_o = \frac{u \cdot \Delta t}{\Delta x}}$$

(Co)

time step

minimum cell value

usually $\underline{C_o < 1}$

Simulation will run in 1-3 but usually results are not correct.

So, we will now set dt value according to the Courant number.

$$\underline{dt = \text{Courant no.} \times dx}$$

we can now increase dx till the results will become const.

1D unsteady Diffusion

$$\frac{d^2\phi}{dx^2}$$

$$u\frac{d\phi}{dx} \rightarrow \text{convection}$$

how fast convection occurs $\rightarrow u$

diffusion coefficient $\rightarrow \Gamma.\frac{d^2\phi}{dx^2} \rightarrow$ diffusion

for thermal, Conduction is diffusion

$$\alpha\frac{d^2T}{dx^2}$$

thermal diffusivity $\rightarrow \alpha$

mass transfer

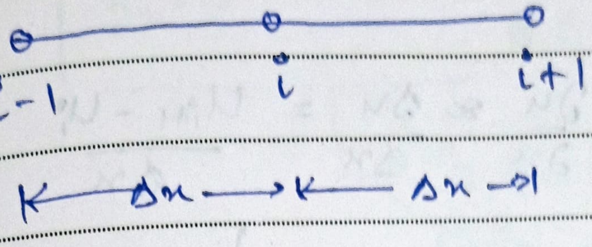
$$\downarrow$$

$$S.\frac{d^2c}{dx^2}$$

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$$

velocity of wave in space

kinematic velocity



$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \approx \frac{\partial}{\partial x}\left(\frac{\Delta u}{\Delta x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{u_{i+1} - u_i}{\Delta x}\right)$$

<u>Laplacian operator.</u>

$$\nabla^2 \phi$$

$$\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)$$

Taylor Series.

prediction in time or space

$$u_{i+1} = u_i + \frac{\Delta x}{1!} \cdot \frac{\partial u}{\partial x}\bigg|_i + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\bigg|_i + \cdots$$

$$u_{i-1} = u_i - \frac{\Delta x}{1!}\frac{\partial u}{\partial x}\bigg|_i + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2}\bigg|_i - \cdots$$

$$\underset{\text{new}}{\boxed{}} = \underset{\text{old}}{\boxed{}} + \underbrace{\left[\boxed{}+\boxed{}+\boxed{}+\boxed{}\cdots\right]}$$

ones which are not significant ignore.

without diffun,

$$\frac{\partial u}{\partial x} \approx \frac{\Delta u}{\Delta x} = \frac{u_{i+1} - u_i}{\Delta x}$$

forward differ

$$u_{i+1} = u_i + \Delta x . \left.\frac{\partial u}{\partial x}\right|_i$$

(Same as Taylor series)

So, backward differ (to predict previous one)

$$u_{i-1} = u_i - \Delta x \left.\frac{\partial u}{\partial x}\right|_i$$

$$\therefore \left.\frac{\partial u}{\partial x}\right|_i = \frac{u_i - u_{i-1}}{\Delta x}$$

So, from Taylor expansion,

(till 2nd differentiation)

from prev. page

$$u_{i+1} + u_{i-1} = 2u_i + \frac{2\Delta x^2}{2!} . \left.\frac{\partial^2 u}{\partial x^2}\right|_i$$

$$\therefore u_{i+1} - 2u_i + u_{i-1} = \Delta x^2 \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \boxed{\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}}$$

↳ Central Difference form approximation for $\frac{\partial^2 u}{\partial x^2}$

if we wanted for $\frac{\partial u}{\partial x}$,

we could have subtracted

$(u_{i+1} - u_{i-1})$

diffusion occurs in all directions equally,

& both forward and backward would matter

So, the discretization eqn will be

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

$$\therefore \; u_i^{n+1} - u_i^n = \Delta t \cdot \nu \cdot \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]$$

$$\therefore \; \boxed{u_i^{n+1} = u_i^n + \nu \cdot \Delta t \cdot \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} \right]}$$

new value = old + Δt [slope]

for diffusion

CFL Condition

$$\Downarrow$$

$$\boxed{C_o = \frac{\nu \, dt}{dx^2} < \frac{1}{2}}$$

for normal cases

$$\frac{u\,dt}{dx}$$

Now, if we run for
100000 times also
(nb = 100000).

we will add a
Convergence Condition

which will check if the
results for 10 consecutive
times are same,
and at that point,
the simulation
can stop.

CFD 2.0

- 1D Convection - Diffusion
- 2D Diffusion
- 2D Convection Diffusion
- 3D Convection Diffusion
- 2D Pressure Gradient
- Navier - Stoke's eqn
- Poisson's eqn

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1D Convection - Diffusion.

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}}$$

Burger's eqⁿ.

Discritization

$$\therefore \frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \left[\frac{u_i^n - u_{i-1}^n}{\Delta x}\right] = \nu \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right]$$

$$\therefore \boxed{u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\underbrace{\left(u_i^n - u_{i-1}^n\right)}_{\text{backward scheme}} + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}$$

Forward scheme $\left(u_{i+1}^n - u_i^n\right)$ also can be used

Code Optimization

a[x : y : z]

start (inclusive) — end (exclusive) — step

e.g.

$$u_i^{n+1} = u_i^n - u_{i-1}^n$$

```
for i in range(1, len(u)):
    print(u[i] - u[i-1])
```

OR

```
print(u[1:] - u[0:-1])
```

To time →

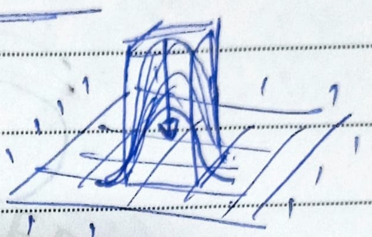
```
import time
t0 = time.time()
// code
t1 = time.time()
dt = t1 - t0
```

2D Diffusion

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}$$

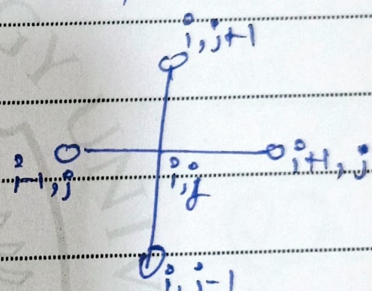
forward diff.

central diff.



from pre

Discretization,



$$\therefore \left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{u^{n+1}_{i,j} - u^{n}_{i,j}}{\Delta t}$$

$$\therefore \left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j}}{\Delta x^2}$$

$$\therefore \left.\frac{\partial^2 u}{\partial y^2}\right|_{i,j} = \frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1}}{\Delta y^2}$$

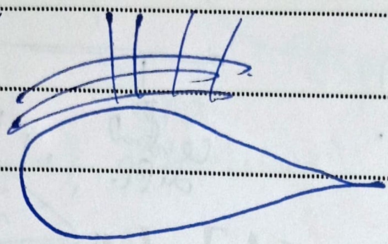
$$\therefore \frac{u^{n+1}_{i,j} - u^{n}_{i,j}}{\Delta t} = \nu\left[\frac{u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j}}{\Delta x^2} + \frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u_{i,j-1}}{\Delta y^2}\right]$$

$$\therefore u^{n+1}_{i,j} = u^{n}_{i,j} + \Delta t \cdot \nu \left[\frac{u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j}}{\Delta x^2} + \frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u_{i,j+1}}{\Delta y^2}\right]$$

For uniform Mesh,

$$\Delta x = \Delta y$$

eg. airfoil



Mesh ~~___~~ has to be kept smaller where high physics and accuracy is required.

In our case,

$$\Delta x = \Delta y$$

So,

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\Delta t \cdot \nu}{\Delta x^2}\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right]$$

fixed value for boundary Condition → Dirichlet boundary Condition.

2D Convection - Diffusion

2D Burger's eqn

$$\cancel{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]}$$

$$\cancel{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]}$$

$$\cancel{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = (1)}$$

$$\cancel{\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (2)}$$

∇

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

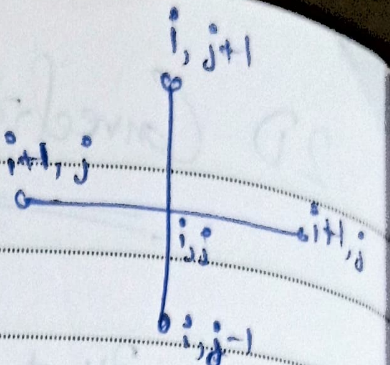
$$\underbrace{\frac{\partial v}{\partial t}}_{\text{unsteady term}} + \underbrace{u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y}}_{\text{divergence term}} = \nu\underbrace{\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]}_{\text{Laplacian term}}$$

2

~~divergence~~

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u\cdot\nabla u$$

$$\therefore \frac{\partial u}{\partial t} = \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}$$



$$\therefore u\frac{\partial u}{\partial x} = u_{i,j}^{n} \cdot \frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x}$$

$$\therefore v\frac{\partial u}{\partial y} = v_{i,j}^{n} \cdot \frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y}$$

$$\therefore \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = \frac{\nu}{\Delta x^2}\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right]$$

explicit eq^n

→ only one term of future and all other are present

(only one unknown)

clear-cut

implicit

→ more terms of future

→ more internal iterations required

Crank Nicholson approach

→ $\frac{1}{2}$ implicit and $\frac{1}{2}$ explicit

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n} \cdot \frac{\left[u_{i+1,j}^{n} - u_{i,j}^{n}\right]}{\Delta x} + v_{i,j}^{n} \frac{\left[u_{i,j+1}^{n} - u_{i,j}^{n}\right]}{\Delta y}$$

$$= \nu \left[\frac{u_{i+1,j}^{n} + u_{i-1,j}^{n} - 2u_{i,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} + u_{i,j-1}^{n} - 2u_{i,j}^{n}}{\Delta y^2} \right]$$

let, Uniform mesh
$(\Delta x = \Delta y)$

$$\therefore u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\Delta t}{\Delta x^2} \Big[\nu \left(u_{i,j+1}^{n} + u_{i,j-1}^{n} + u_{i+1,j}^{n} + u_{i-1,j}^{n} - 4u_{i,j}^{n} \right) - \Delta x \left\{ u_{i,j}^{n} \left(u_{i+1,j}^{n} - u_{i,j}^{n} \right) + v_{i,j}^{n} \left(u_{i,j+1}^{n} - u_{i,j}^{n} \right) \right\} \Big]$$

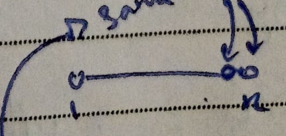
Similarly, 2nd eq^n,

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \frac{\Delta t}{\Delta x^2} \Big[\nu \left(v_{i+1,j}^{n} + v_{i-1,j}^{n} + v_{i,j+1}^{n} + v_{i,j-1}^{n} - 4v_{i,j}^{n} \right) - \Delta x \left\{ u_{i,j}^{n} \left(v_{i+1,j}^{n} - v_{i,j}^{n} \right) + v_{i,j}^{n} \left(v_{i,j+1}^{n} - v_{i,j}^{n} \right) \right\} \Big]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right]$$

for boundary, we can give Neumann boundary condition

$$\frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial x} = 0 \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = 0$$



$$\frac{u_n - u_{n-1}}{\Delta x} = 0$$

3D - Convection Diffusion

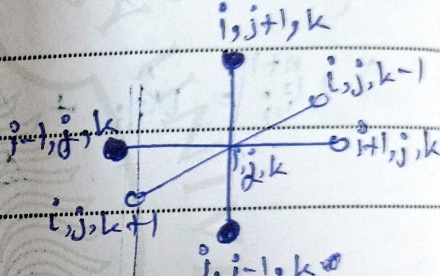
[Burger's eqⁿ]

① $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

② $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

③ $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Applying discretization,



$$\frac{\partial u}{\partial t} = \frac{u^{n+1}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t}$$

$$\frac{\partial u}{\partial x} = \frac{u^{n}_{i+1,j,k} - u^{n}_{i,j,k}}{\Delta x}$$

$$\frac{\partial u}{\partial y} = \frac{u^{n}_{i,j+1,k} - u^{n}_{i,j,k}}{\Delta y}$$

$$\frac{\partial u}{\partial z} = \frac{u^{n}_{i,j,k+1} - u^{n}_{i,j,k}}{\Delta z}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u^{n}_{i+1,j,k} - 2u^{n}_{i,j,k} + u^{n}_{i-1,j,k}}{\Delta x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u^{n}_{i,j+1,k} - 2u^{n}_{i,j,k} + u^{n}_{i,j-1,k}}{\Delta y^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{u^{n}_{i,j,k+1} - 2u^{n}_{i,j,k} + u^{n}_{i,j,k-1}}{\Delta z^2}$$

So, for 1st eq^n.

$$\frac{u^{n+1}_{i,j,k} - u^{n}_{i,j,k}}{\Delta t} + u^{n}_{i,j,k}\frac{\left(u^{n}_{i+1,j,k} - u^{n}_{i,j,k}\right)}{\Delta x} + v^{n}_{i,j,k}\frac{\left(u^{n}_{i,j+1,k} - u^{n}_{i,j,k}\right)}{\Delta y}$$

$$+ w^{n}_{i,j,k}\frac{\left(u^{n}_{i,j,k+1} - u^{n}_{i,j,k}\right)}{\Delta z}$$

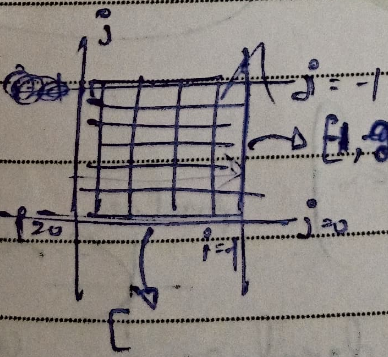
$$= \nu\left[\frac{u^{n}_{i+1,j,k} - 2u^{n}_{i,j,k} + u^{n}_{i-1,j,k}}{\Delta x^2} + \frac{u^{n}_{i,j+1,k} - 2u^{n}_{i,j,k} + u^{n}_{i,j-1,k}}{\Delta y^2} + \frac{u^{n}_{i,j,k+1} - 2u^{n}_{i,j,k} + u^{n}_{i,j,k-1}}{\Delta z^2}\right]$$

let uniform mesh

$(\Delta x = \Delta y = \Delta z)$

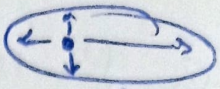
$$\therefore u^{n+1}_{i,j,k} = u^{n}_{i,j,k} + \frac{\Delta t}{\Delta x^2}\Big[\nu\left(u^{n}_{i+1,j,k} + u^{n}_{i-1,j,k} + u^{n}_{i,j+1,k} + u^{n}_{i,j-1,k} + u^{n}_{i,j,k+1} + u^{n}_{i,j,k-1} - 6u^{n}_{i,j,k}\right)$$

$$- \Delta x\left\{u^{n}_{i,j,k}\left(u^{n}_{i+1,j,k} - u^{n}_{i,j,k}\right) + v^{n}_{i,j,k}\left(u^{n}_{i,j+1,k} - u^{n}_{i,j,k}\right) + w^{n}_{i,j,k}\left(u^{n}_{i,j,k+1} - u^{n}_{i,j,k}\right)\right\}\Big]$$



Similarly the eq^n can be re-written for v and w

2D



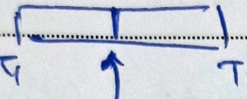
elliptic PDE

(Conduction)
(Diffusion)

ellipse
(closed from all sides)
⇓
all boundary conditions affect the slope.

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

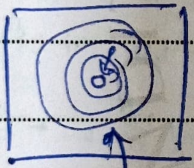


T here will depend on both sides.

Laplacian eqn

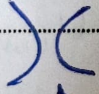
hyperbolic PDE Condition

→ propagation in all directions Equally



ripple properties in water will not depend on boundary.

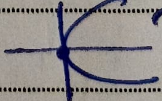
$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$



hyperbola.

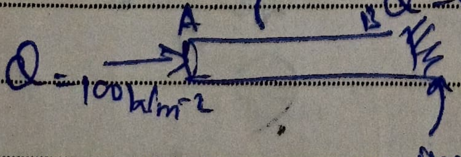
Parabolic PDE Condition

parabola

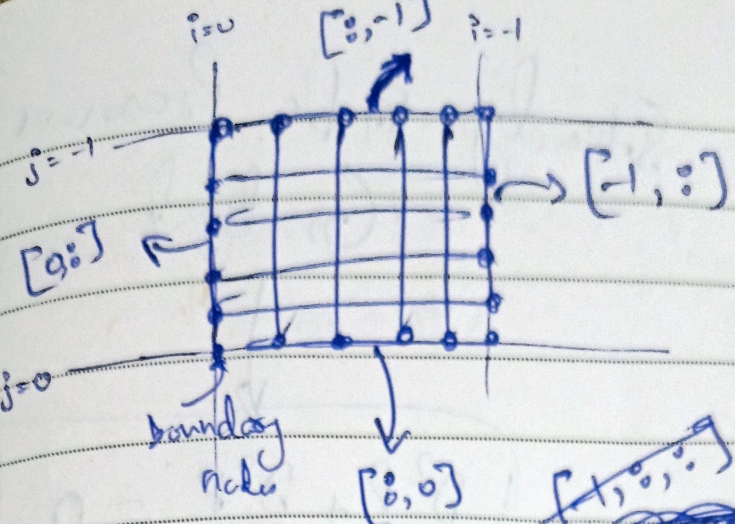


→ open from one side (only one of boundary side affects)

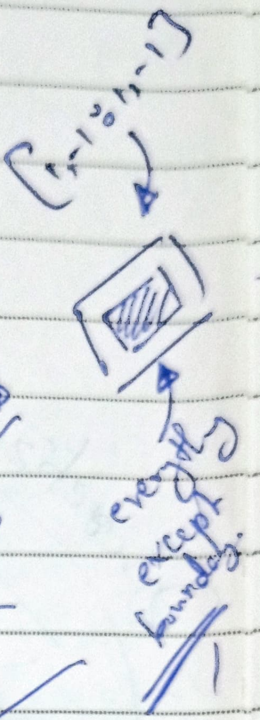
→ heat flow only depends on side A.



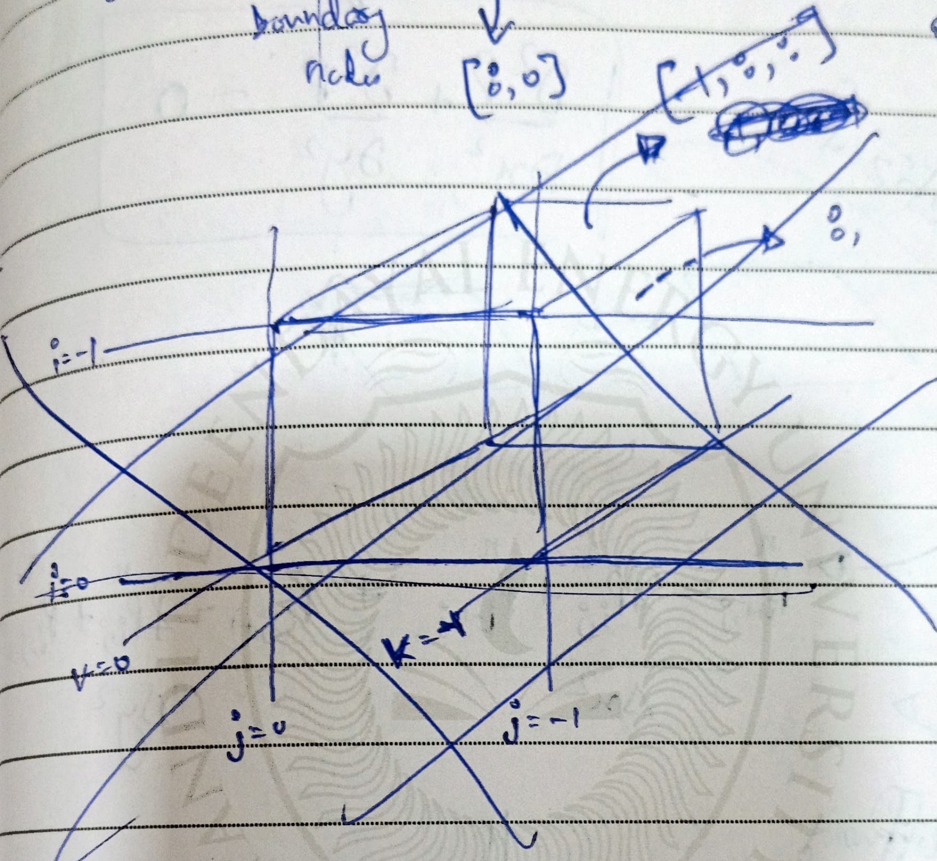
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$



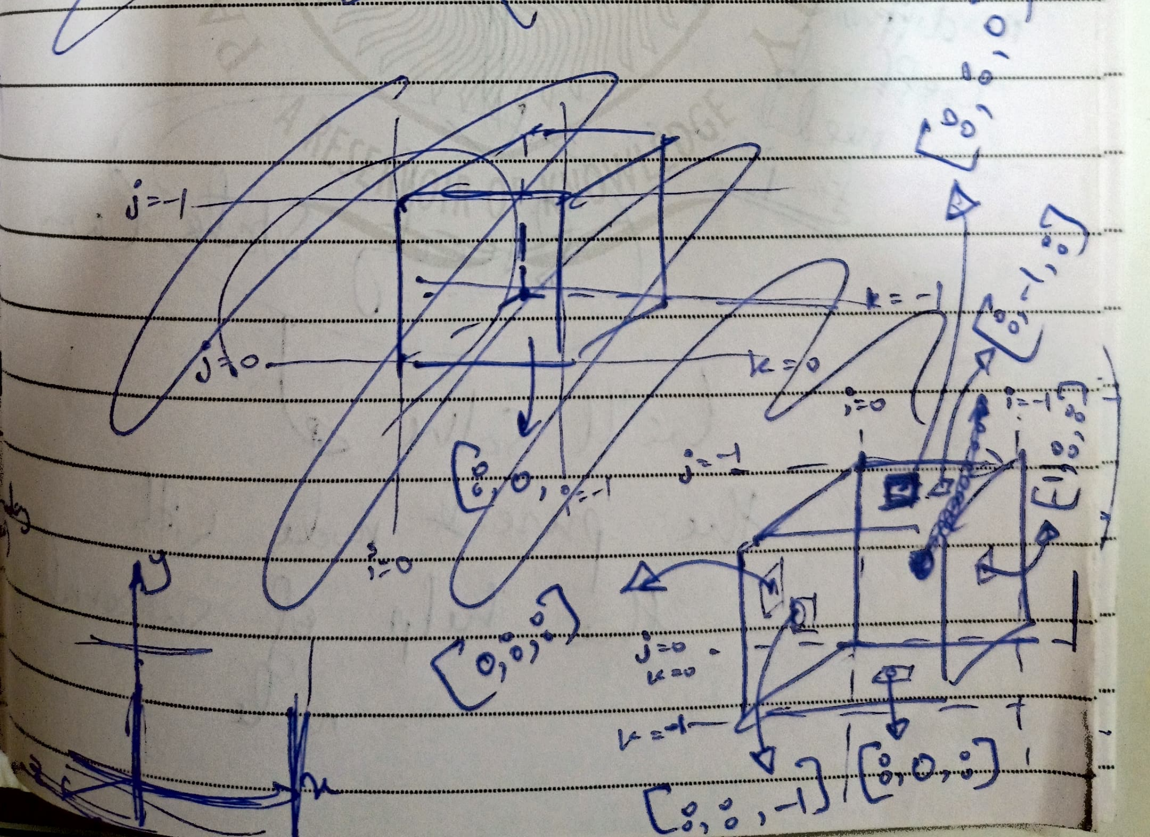
i=0
[:, -1]
i=-1
j=-1
[-1, :]
[0, :]
j=0
boundary nodes
[:, 0]



[1:-1, 1:-1]
everything except boundary



[-1, :, :]
i=-1
i=0
k=0
k=-1
j=0
j=-1

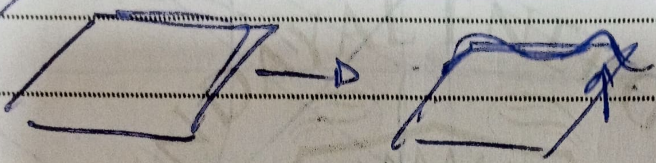


j=-1
j=0
k=-1
k=0
[:, :, 0]
[:, -1, :]
i=0
i=-1
[-1, :, :]
[0, :, :]
j=0
k=0
k=-1
[:, :, -1]
[:, 0, :]
y
x

Steady State Pressure Gradient

$$\boxed{\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0}$$

Discretization



$$\frac{P^n_{i+1,j} - 2P^n_{i,j} + P^n_{i-1,j}}{\Delta x^2} + \frac{P^n_{i,j+1} - 2P^n_{i,j} + P^n_{i,j-1}}{\Delta y^2} =$$

no diffusion coefficient needed for pressure.

Steady state case

We'll solve the present node with the help of surrounding node

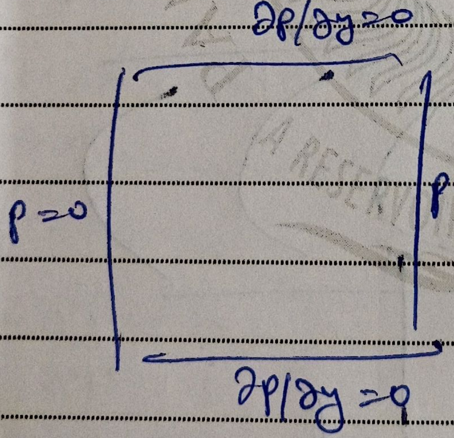
$$P^n_{i,j} = \frac{\Delta y^2\left(P^n_{i+1,j}+P^n_{i-1,j}\right)+\Delta x^2\left(P^n_{i,j+1}+P^n_{i,j-1}\right)}{2\left(\Delta x^2+\Delta y^2\right)}$$

Boundary Condition

→ $p=0$ at $x=0$

$p=y$ at $x=2$

$\frac{\partial p}{\partial y}=0$ at $y=0,1$



we'll just get one constant gradient

in steady state →

(NOTE) → it is like guessing game, we are not iterating in ~~time~~ but we are just guessing more times to get final solution fast

Boundary Condition directly assigned a const.

↓

Dirichlet Boundary Condition

Something like a gradient

↓

Neumann

↓

Same value with a layer inside.

↓

Mixed

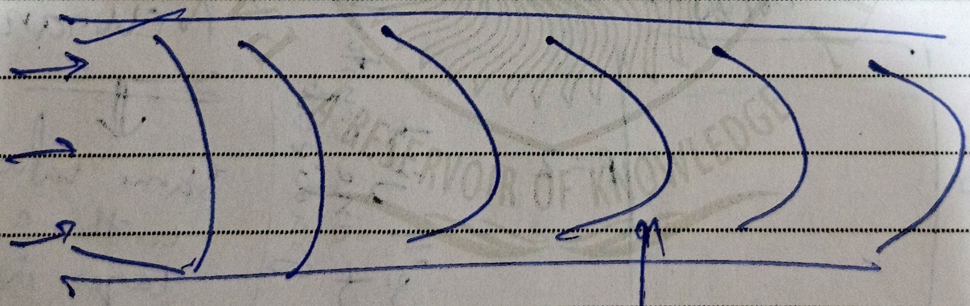
↓

Robin Condition

#pro tip

If we have less computing power

we first solve a problem in steady state condition and then feed the solution as ~~the unsteady in~~ problem for unsteady simulation



steady

$$\boxed{\frac{\partial \vec{v}}{\partial t} + \vec{v}(\vec{v}\cdot\nabla) = \nu \nabla^2 \vec{v} - \frac{1}{\rho}\nabla p}$$

Navier Stoke's eq^n

~~$\nabla\cdot\vec{v} = 0$~~

Prof Suhas Patankar, Spalding

SIMPLE algo.

Semi Implicit Method for pressure linked eq^{n}s

steady

for unsteady

PISO algo.

Pressure implicit with splitting of operators

Laplacian eq^n on pressure

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$$

without deformation, uniform gradient (conserved)

Poisson's eq^n

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

deformation will be there.

but, fluid cannot be permanently deformed.

shear stress, the fluid will make it move.

algo. to solve when 2 eq^{n}s, 3 unknowns.

$p \rightarrow u,v \rightarrow p \rightarrow u,v \rightarrow \ldots$

semi implicit

pressure known, u,v,w predicted known

guessing in first

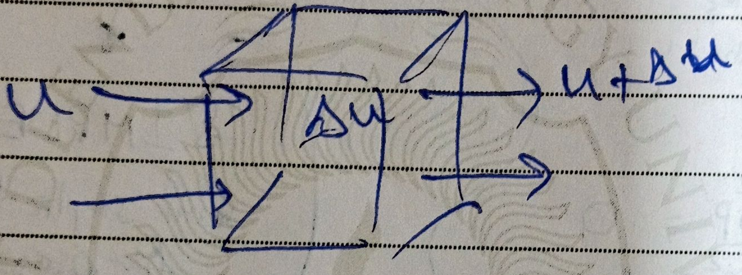
Continuity eq^n

$$\nabla \cdot \vec{v} = 0$$

also says that flow is incompressible or not

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

flux will be conserved.



entering part and leaving part is conserved

1st law of Thermodynamics.

along with Navier-stoke's eq^n, we'll solve this eq^n always, this ensures that real-life like things are there

Original Continuity eq^n

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\vec{v}) = 0$$

$$\Downarrow$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

There is a density variation for compressible fluids.

Navier-Stoke's eq^n

x-momentum eq^n and y-momentum eq^n and z-momentum eq^n

momentum of incompressible flow, density term is multiplied.

$m \times v$ is momentum.

on next page.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

2 eq^{n}s, 3 unknowns.

$\hookrightarrow v, u, p$

Taking Derivatives of both eq^{n}s,

$$\cancel{\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)} + \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) + \cancel{\frac{\partial}{\partial x}\left[\nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]}$$

$$\cancel{\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)} + \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(v\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left[\nu\cancel{\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)}\right]$$

Pressure does depend on time

Consistency

~~will~~

~~We'll ignoring temporal (time-dependent) terms because pressure will not depend on time.~~

Can these

pressure depends only on spatial terms.

ignoring 2nd order derivatives as very small

Reasonable approximation

& both the eqⁿ's become,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}+u\frac{\partial^2 u}{\partial x^2}+\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}+v\frac{\partial^2 u}{\partial x\partial y}=\frac{-1}{\rho}\cdot\frac{\partial^2 p}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y}+v\frac{\partial^2 v}{\partial y^2}=\frac{-1}{\rho}\frac{\partial^2 p}{\partial y^2}$$

Now, adding both the eqⁿ's,

$$\frac{-1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{2\partial u\partial v}{\partial x\partial y}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right)+\cancel{u\frac{\partial^2 v}{\partial x\partial y}}+\cancel{v\frac{\partial^2 u}{\partial x\partial y}}+\cancel{u\frac{\partial^2 u}{\partial x^2}}+\cancel{v\frac{\partial^2 v}{\partial y^2}}$$

non linearity with higher order derivative can be neglected.

these terms become zero because $\frac{\partial v}{\partial x}=0 \rightarrow v$ does not depend on x and similarly, $\frac{\partial u}{\partial y}=0$

thus almost converges to zero → continuity eqⁿ

$$\boxed{\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=-\rho\left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{2\partial u\cdot\partial v}{\partial x\cdot\partial y}\right]+\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$$

→ Poisson's eqⁿ

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}=b$$

Central Diff. on all → for consistency.

This is zero because of continuity but should inherently become zero. But will never become.

Q)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$ 1D Convection

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2}$$ 1D diffusion

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 \\ \text{and, } &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 \end{aligned}\right]$$ 2D convection

$$\left.\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}\right]$$ 2D diffusion

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2} \end{aligned}\right]$$ 2D Burger's

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \xrightarrow{\text{[illegible]}} \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \\ &\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left\{\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{2\,\partial u\,\partial v}{\partial x\,\partial y}\right) + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\} \end{aligned}\right]$$ 2D Navier-Stokes's eqn

Poisson's eqn.

CFD 3.0

→ 2D Cavity
→ 2D Channel

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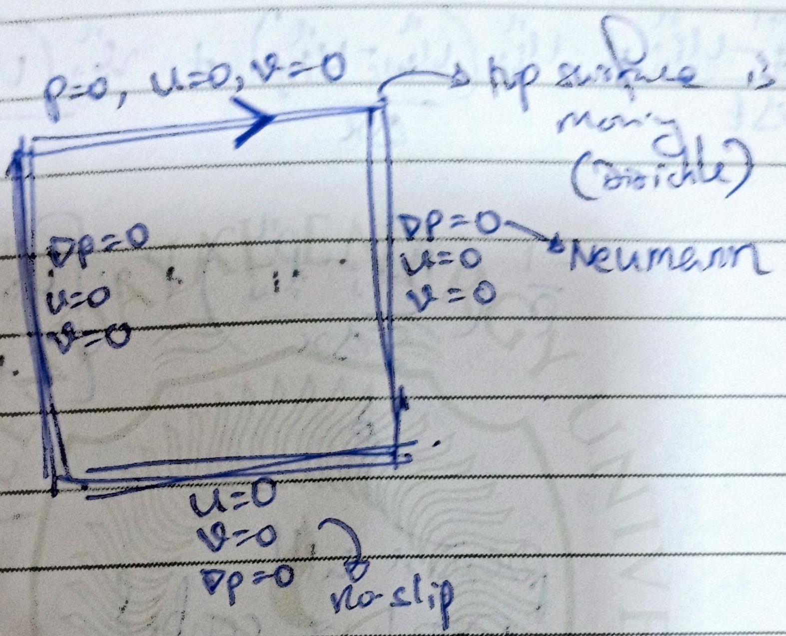
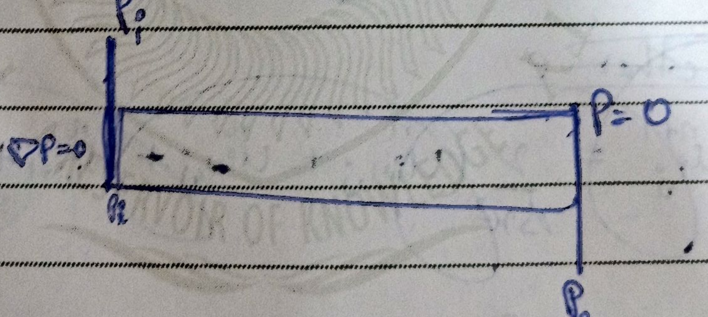
For the cavity,

Eqⁿs for 2D



pressure gradient is required because, for flow



pressure unknown

⇓

u, v is initialized

So, we find p

u,v → p → u,v → p → ...

then from it, new u,v

instead, we are using PISO algo.

Discretizing the Navier-Stoke's eqn

$\frac{\partial u}{\partial t} = \frac{u_{i}^{n+1} - u_{i}^{n}}{\Delta t}$

$\frac{\partial u}{\partial x}$

let, $\Delta x = \Delta y$

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right)}{\Delta x} + v_{i,j}^{n}\frac{\left(u_{i,j+1}^{n} - u_{i,j}^{n}\right)}{\Delta x}$$

$$= \frac{-1}{\rho}\left(\frac{p_{i+1,j}^{n} - p_{i-1,j}^{n}}{2\Delta x}\right) + \nu\frac{\left[u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right]}{\Delta x^2}$$

pressure will have central difference.

~~for other~~ e

$$\therefore u_{i,j}^{n+1} = \frac{\nu}{\Delta x^2}\left(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right)$$

$$\therefore u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t\left[\frac{\nu}{\Delta x^2}\left(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right) - \frac{1}{\Delta x}\left\{\frac{1}{\rho}\frac{\left(p_{i+1,j}^{n} - p_{i-1,j}^{n}\right)}{2} + u_{i,j}^{n}\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right) + v_{i,j}^{n}\left(u_{i,j+1}^{n} - u_{i,j}^{n}\right)\right\}\right]$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \frac{\Delta t}{\Delta x^2}\Big[\nu \left(v_{i+1,j}^n + v_{i-1,j}^n + v_{i,j+1}^n + v_{i,j-1}^n - 4v_{i,j}^n \right) - \Delta x \Big\{ \frac{1}{\rho} \frac{\left(p_{i,j+1}^n - p_{i,j-1}^n\right)}{2} + u_{i,j}^n \left(v_{i+1,j}^n - v_{i,j}^n \right) + v_{i,j}^n \left(v_{i,j+1} - v_{i,j}^n \right) \Big\} \Big]$$

for poisson's eq^n,

$$\frac{\left(p_{i+1,j}^n + p_{i-1,j}^n - 2p_{i,j}^n\right)}{\Delta x^2} + \frac{\left(p_{i,j+1}^n + p_{i,j-1}^n - 2p_{i,j}^n\right)}{\Delta y^2} =$$

$$+ \rho \Bigg[\frac{1}{\Delta t} \left\{ \frac{\left(u_{i,j+1} - u_{i,j-1}\right)}{2\Delta x} + \frac{\left(v_{i+1,j} - v_{i-1,j}\right)}{2\Delta y} \right\}$$

$$- \Bigg\{ \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \cdot \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \cdot \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}$$

$$+ 2 \cdot \frac{\left(u_{i+1,j} - u_{i-1,j}\right)}{2\Delta x} \frac{\left(v_{i,j+1} - v_{i,j}\right)}{2\Delta y}$$

Central difference

in discretized eq^n, we include the time term as pressure is steady but needs to change with time.

if in above eqⁿ;

$$RHS = b_{i,j}$$

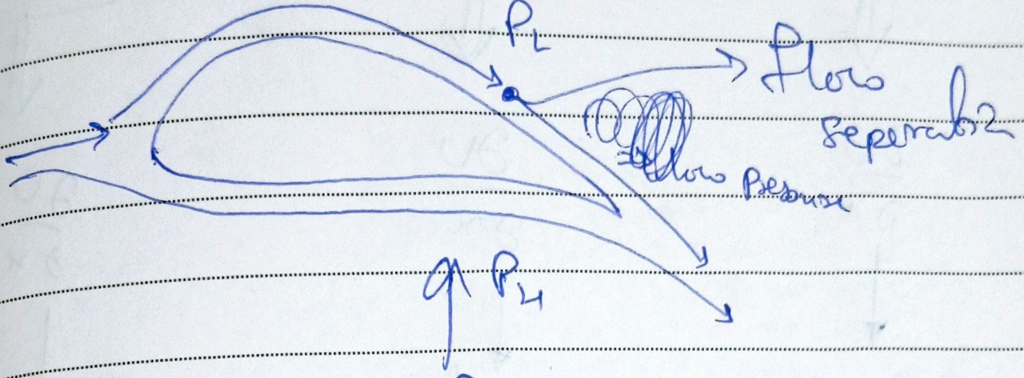
and, $\Delta x = \Delta y$

$$p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j+1} + p^n_{i,j-1} - 4p^n_{i,j} = \Delta x^2 b$$

$$\therefore \; p^n_{i,j} = \frac{p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j+1} + p^n_{i,j-1} - \Delta x^2 b}{4}$$

$$\therefore \; p^n_{i,j} = \frac{p^n_{i+1,j} + p^n_{i-1,j} + p^n_{i,j+1} + p^n_{i,j-1}}{4} - \frac{\Delta x \rho}{8} \Bigg[\frac{1}{\Delta t}\left(u_{i+1,j} - u_{i-1,j} + v_{i,j+1} - v_{i,j-1}\right)$$

$$- \frac{1}{\Delta x}\left\{ \left(u_{i+1,j} - u_{i-1,j}\right)^2 + \left(v_{i,j+1} - v_{i,j-1}\right)^2 + \left(u_{i,j+1} - u_{i,j-1}\right)\left(v_{i+1,j} - v_{i-1,j}\right) \right\} \Bigg]$$



Lift

in race cars
→ spoilers : to reduce drag
rear wings : down force

→ stability
→ Convergence
→ Consistency
→ forward diff., backward diff., central diff.

→ CFD Online
→ forum

→ CSI Openfoam
→ openfoam.org

Finite difference method

$\Downarrow$

$\frac{\partial U}{\partial x}$

$\downarrow$

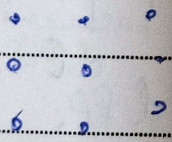
Taylor Series

$\downarrow$

Algebraic eqn

$\downarrow$

$\frac{U_{i+1}-U_{i-1}}{2\Delta x}$



[node]

$\downarrow$

for simple cases.

Finite Volume method

$\Downarrow$

$\frac{\partial U}{\partial x}$

$\downarrow$

integral

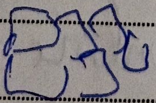
$\int_{CV} \frac{\partial U}{\partial x} dV$

$\downarrow$

Divergence theo.

(algebraic)

$\int_A n U dA$



[Volumes]

Finite element method

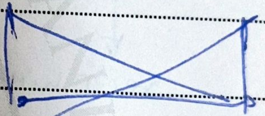
$\Downarrow$

$\frac{\partial U}{\partial x}$

$\downarrow$

basis func

$U = U_G \cdot \Phi_G + U_t \cdot \Phi_t$



$\downarrow$

$\frac{\partial U}{\partial x} = U_G \frac{\partial \Phi_G}{\partial x} + U_t \frac{\partial \Phi_t}{\partial x}$

[Cells]

No physics

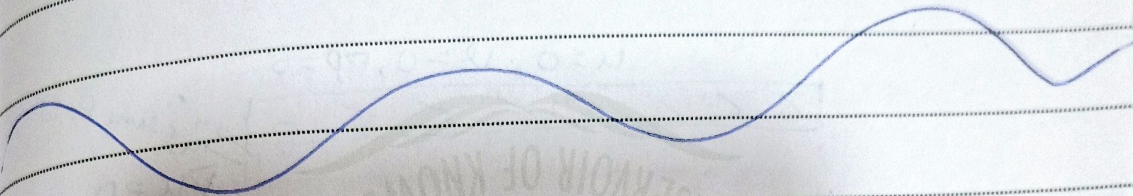
$\downarrow$

Structural

$\frac{\partial u}{\partial x} + \frac{\partial U}{\partial x}$ cannot become zero sometimes so we use this method.

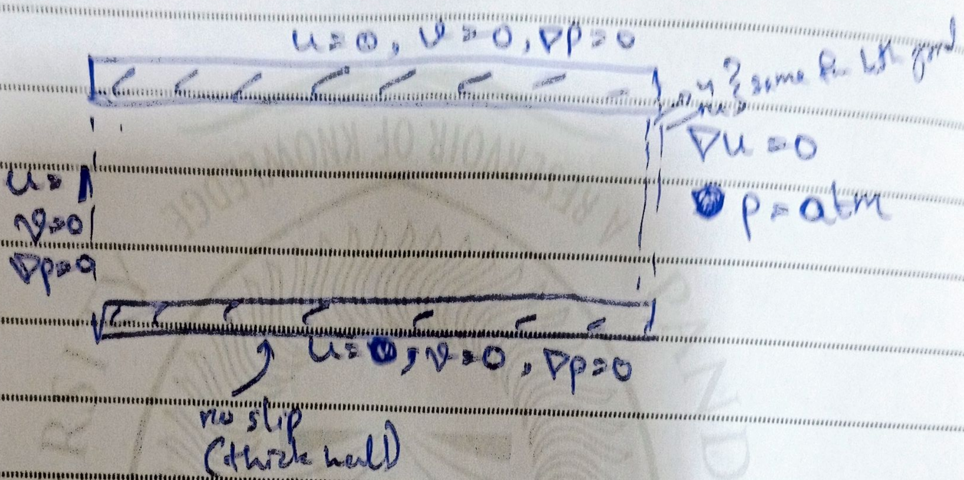
Reynold's no.

$$R_e = \frac{\rho v D}{\mu} = \frac{v D}{\nu}$$



2D Channel

1 → problem and domain
2 → boundary condition



top-bottom can also have periodic boundary condition.

to include gravity → buoyancy has to be included as source term for y momentum eqn.

$$-g\beta\Delta T$$

$\Delta T = T - T_{ref}$

→ Boussinesq approximation

$u, v \rightarrow T \rightarrow u, v \rightarrow \ldots$

no gravity but heat transfer → $u, v \rightarrow T$ (one-way coupling)

One more governing eqn. ←

(s phase)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

Book

[libgen.is]

- Anderson CFD and heat transfer → 1st book
- S. Patankar
- Malala Sekhera
- Moukaled Darwish, CFD (FVM)

Source term → force → to minimize effect of pressure driven flow.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + F$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial v}{\partial y} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\right\}$$

(RHS = b)

Discretization

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\cdot\frac{\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right)}{\Delta x} + v_{i,j}^{n}\frac{\left(u_{i,j+1}^{n} - u_{i,j}^{n}\right)}{\Delta y}$$

$$= -\frac{1}{\rho}\frac{\left(p_{i+1,j}^{n} - p_{i-1,j}^{n}\right)}{2\Delta x} + \nu\left(\frac{u_{i+1,j}^{n} + u_{i-1,j}^{n} - 2u_{i,j}^{n}}{\Delta x^2} + \frac{v_{i,j+1}^{n} + v_{i,j-1}^{n} - 2v_{i,j}^{n}}{\Delta y^2}\right)$$

$$+ F_{i,j}$$

let, $(\Delta x = \Delta y)$

$$\therefore u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t \Bigg[\frac{\nu}{\Delta x^2}\left(u_{i+1,j}^{n} + u_{i-1,j}^{n} + u_{i,j+1}^{n} + u_{i,j-1}^{n} - 4u_{i,j}^{n}\right)$$

$$- \frac{1}{\rho}\left(\frac{p_{i+1,j}^{n} - p_{i-1,j}^{n}}{2\Delta x}\right) - u_{i,j}^{n}\frac{\left(u_{i+1,j}^{n} - u_{i,j}^{n}\right)}{\Delta x}$$

$$- v_{i,j}^{n}\frac{\left(u_{i,j+1}^{n} - u_{i,j}^{n}\right)}{\Delta x} + F_{i,j}\Bigg]$$

1st eqn

$$v_{i,j}^{n+1} = v_{i,j}^{n} + \Delta t \Bigg[\frac{\nu}{\Delta x}\left(v_{i+1,j}^{n} + v_{i-1,j}^{n} + v_{i,j+1}^{n} + v_{i,j-1}^{n} - 4v_{i,j}^{n}\right)$$

$$- \frac{1}{\rho}\left(\frac{p_{i,j+1}^{n} - p_{i,j-1}^{n}}{2\Delta y}\right) - u_{i,j}^{n}\frac{\left(v_{i+1,j}^{n} - v_{i,j}^{n}\right)}{\Delta x}$$

(Δy)

$$- v_{i,j}^{n}\frac{\left(v_{i,j+1}^{n} - v_{i,j}^{n}\right)}{\Delta x}\Bigg]$$

2nd eqn

$$\frac{\left(p_{i+1,j}^{n} + p_{i-1,j}^{n} - 2p_{i,j}^{n}\right)}{\Delta x} + \frac{\left(p_{i,j+1}^{n} + p_{i,j-1}^{n} - 2p_{i,j}^{n}\right)}{\Delta y}$$

$$= -\rho \Bigg[\left(\frac{u_{i+1,j}^{n} - u_{i-1,j}^{n}}{2\Delta x}\right)^2 + \left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta x}\right)^2 - 2\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right)\left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta x}\right)$$

$$+ \frac{1}{\Delta t}\left\{\left(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta x}\right) + \left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}\right)\right\}\Bigg]$$

3rd eqn